



TITLE:

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CITATION:

HIROSHIMA, FUMIO. Strong and Weak Coupling Limits of Interaction Models of Quantum Fields and Particles. 数理解析研究所講究録 1997, 1013: 207-224

ISSUE DATE:

1997-09

URL:

<http://hdl.handle.net/2433/61557>

RIGHT:

Strong and Weak Coupling Limits of Interaction Models of Quantum Fields and Particles

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1 INTRODUCTION

Asymptotic behaviors of scaling Hamiltonians which describe interactions of particles and quantized fields are considered. In a mathematical formulation, interaction Hamiltonians of the particles and the quantized fields are described by the theory of self-adjoint operators acting in the tensor product of two Hilbert spaces over the complex field \mathbb{C} . Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces. We define a self-adjoint operator \mathbf{H} acting in the tensor product of \mathcal{H}_1 and \mathcal{H}_2 , $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, by

$$\mathbf{H} = H_1 \otimes I + \alpha H_{int} + I \otimes H_2.$$

Here H_1 and H_2 are self-adjoint operators in \mathcal{H}_1 and \mathcal{H}_2 , respectively, H_{int} is a symmetric operator in \mathcal{H} and $\alpha \in \mathbb{R}$ is a coupling constant. Then, for the given self-adjoint operator \mathbf{H} , we define “ β -coupling Hamiltonian, $\mathbf{H}_\beta(\Lambda)$ ”, by

$$\mathbf{H}_\beta(\Lambda) = H_1 \otimes I + \Lambda \alpha H_{int} + \Lambda^\beta I \otimes H_2, \quad 1 \leq \beta. \quad (1.1)$$

Introducing a renormalization $E_\beta(\Lambda)$ which goes to infinity or minus infinity as $\Lambda \rightarrow \infty$ in some sense, we want to investigate the following asymptotic behaviors

$$s - \lim_{\Lambda \rightarrow \infty} e^{-it(\mathbf{H}_\beta(\Lambda) - E_\beta(\Lambda))} = \mathbf{U} \left(e^{-itH_{eff}} \otimes \mathbf{P} \right) \mathbf{U}^{-1}, \quad t \in \mathbb{R}. \quad (1.2)$$

Here H_{eff} is a self-adjoint operator in \mathcal{H}_1 , which is called “effective Hamiltonian”, \mathbf{U} is a unitary operator in \mathcal{H} and \mathbf{P} a projection operator onto a one-dimensional subspace in \mathcal{H}_2 . It

seems to be useful to readers to collect some background ingredient. Motivation of this paper is [1] and [3]. In [1], in order to give an interpretation of a physical phenomenon “Lamb shift” without formal perturbation theory, A.Arai elaborates a scaling limit of the Pauli-Fierz model. The scaling limit corresponds to the case $\beta = 1$ in (1.1). In [3], E.B.Davies studies a scaling limit of the Nelson model to derive a Schrödinger Hamiltonian (effective Hamiltonian) with a scalar potential. The scaling limit corresponds to the case $\beta = 2$ in (1.1). In this paper, we deal with the Nelson model [2,3,4,5,8,10], the Pauli-Fierz model [1,6,7,8,9,10] and the spin-boson model [1]. Thus considering scaling limits as in (1.2) for these models is an extension of those considered in [1,2,3,6,7,10]. We organize this paper as follows. In section 2, we overview an abstract theory of a scaling limit of self-adjoint operators. In section 3,4 and 5, we study the Nelson model, the Pauli-Fierz model and the spin-boson model, respectively. In section 6, we give some remarks.

2 FUNDAMENTAL FACTS

2.1 An abstract Boson Fock space

In this subsection we define an abstract Boson Fock space and basic notations. For a separable Hilbert space \mathcal{H} over \mathbb{C} , we denote the scalar product by $\langle f, g \rangle_{\mathcal{H}}$ and the associated norm by $\|f\|_{\mathcal{H}}$, where the scalar product is linear in g and antilinear in f . For the tempered distributions f and g , the notation \bar{f} denotes the complex conjugate of f , and \hat{f} (resp. \check{g}) the Fourier transform of f (resp. the inverse Fourier transform of g). We denote the domain of an operator A by $D(A)$. The Boson Fock space over the Hilbert space \mathcal{H} is defined by

$$\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} [\otimes_s^n \mathcal{H}],$$

where $\otimes_s^n \mathcal{H}$, $n \geq 1$, denotes the n -fold symmetric tensor product of \mathcal{H} , and $\otimes_s^0 \mathcal{H} = \mathbb{C}$. Define $\Omega_{\mathcal{H}} = \{1, 0, 0, \dots\}$. Let the annihilation and creation operators in the Boson Fock space denoted by $a_{\mathcal{H}}(f)$, $f \in \mathcal{H}$ and $a_{\mathcal{H}}^{\dagger}(g)$, $g \in \mathcal{H}$, respectively. It is well known that

$$\mathcal{F}_{\mathcal{H}}^{\infty} \equiv \mathbf{L} \left\{ a_{\mathcal{H}}^{\dagger}(f_1) \dots a_{\mathcal{H}}^{\dagger}(f_n) \Omega, \Omega | f_j \in \mathcal{H}, j = 1, \dots, n, n \geq 1 \right\}$$

is dense in $\mathcal{F}_{\mathcal{H}}$, where \mathbf{L} denotes the linear hull of the vectors in $\{\dots\}$. The annihilation and the creation operators in the Boson Fock space satisfy the following canonical commutation

relations on $\mathcal{F}_{\mathcal{H}}^{\infty}$:

$$\begin{aligned} [a_{\mathcal{H}}(f), a_{\mathcal{H}}^{\dagger}(g)] &= \langle \bar{f}, g \rangle_{\mathcal{H}}, \\ [a_{\mathcal{H}}^{\sharp}(f), a_{\mathcal{H}}^{\sharp}(g)] &= 0, \end{aligned}$$

where $a_{\mathcal{H}}^{\sharp}$ means $a_{\mathcal{H}}$ or $a_{\mathcal{H}}^{\dagger}$. Let h be a self-adjoint operator in \mathcal{H} . Define $d\Gamma_{\mathcal{H}}(h)$ by

$$\begin{aligned} d\Gamma_{\mathcal{H}}(h)\Omega &= 0, \\ d\Gamma_{\mathcal{H}}(h)a_{\mathcal{H}}^{\dagger}(f_1)\dots a_{\mathcal{H}}^{\dagger}(f_n)\Omega_{\mathcal{H}} &= \sum_{j=1}^n a_{\mathcal{H}}^{\dagger}(f_1)\dots a_{\mathcal{H}}^{\dagger}(hf_j)\dots a_{\mathcal{H}}^{\dagger}(f_n)\Omega_{\mathcal{H}}, \quad f_j \in D(h). \end{aligned}$$

Then $d\Gamma_{\mathcal{H}}(h)$ is essentially self-adjoint. Let us use the same notation as $d\Gamma_{\mathcal{H}}(h)$ for its self-adjoint extension.

2.2 An abstract theory of a scaling limit

We overview an abstract theory of a scaling limit of self-adjoint operators acting in a tensor product Hilbert space established in [1] with a little modification. Let \mathcal{K} be a Hilbert space and put $\mathcal{X} = \mathcal{H} \otimes \mathcal{K}$. Suppose that an operator, A (resp. B), is a nonnegative self-adjoint operator in \mathcal{H} (resp. \mathcal{K}) and $\text{Ker} B = \{kG | k \in \mathbb{C}, \|G\|_{\mathcal{X}} = 1\}$. Set the projection operator onto $\text{Ker} B$ by P_B . We suppose that a family of self-adjoint operators, $\{C_{\Lambda}\}_{\Lambda>0}$, in \mathcal{X} admits the following conditions:

- (1) For any $\epsilon > 0$, there exists Λ_0 so that, for all $\Lambda > \Lambda_0$, $D(C_{\Lambda}) \supset D(A \otimes I + \Lambda I \otimes B)$ with

$$\|C_{\Lambda}\Phi\|_{\mathcal{X}} \leq \epsilon \|(A \otimes I + \Lambda I \otimes B)\Phi\| + b(\epsilon)\|\Phi\|_{\mathcal{X}}, \quad \Phi \in D(A \otimes I) \cap D(I \otimes B),$$

where $b(\epsilon) > 0$ is a constant independent of $\Lambda > \Lambda_0$.

- (2) There exists a symmetric operator C in \mathcal{X} so that $D(C) \supset D(A) \otimes \text{Ker} B$ and, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s - \lim_{\Lambda \rightarrow \infty} C_{\Lambda}(A \otimes I + \Lambda I \otimes B - z)^{-1} = C \{(A - z)^{-1} \otimes P_B\}.$$

We define an operator $E_G(C)$ with the domain $D(E_G(C)) = D(A)$ by

$$\langle f, E_G(C)g \rangle_{\mathcal{H}} = \langle f \otimes G, C(g \otimes G) \rangle_{\mathcal{X}}, \quad f \in \mathcal{H}, g \in D(A).$$

We call $E_G(C)$ “the partial expectation of C with respect to G ”. Set

$$K_{eff} = A + E_G(C).$$

The following proposition is fundamental in this paper.

Proposition 2.1 ([1, Theorem 2.1]) *Let operators A, B, C_Λ , and C be as above. Then*

(1) *For $\Lambda > \Lambda_0$, $K_\Lambda = A \otimes I + \Lambda I \otimes B + C_\Lambda$ is self-adjoint on $D(A \otimes I) \cap D(I \otimes B)$ and uniformly bounded from below. Moreover $E_G(C)$ is infinitesimally small with respect to A , i.e., K_{eff} is self-adjoint on $D(A)$.*

(2) *For $z \in \mathbb{C} \setminus \mathbb{R}$*

$$s - \lim_{\Lambda \rightarrow \infty} (K_\Lambda - z)^{-1} = (K_{eff} - z)^{-1} \otimes P_B. \quad (2.1)$$

Finally we note a fundamental fact.

Proposition 2.2 *Let K_Λ and K_{eff} satisfy (2.1). Then*

$$s - \lim_{\Lambda \rightarrow \infty} e^{-itK_\Lambda} = e^{-itK_{eff}} \otimes P_B.$$

Proof: See [1, Theorem 2.2] □

By Proposition 2.2, it is enough to show strong resolvent limits of β -coupling Hamiltonian to investigate (1.2).

3 THE NELSON MODEL

3.1 The Nelson model

In this section, we consider the Nelson Hamiltonian with an ultraviolet cut-off function $\hat{\varrho}$ and with a finite number of nonrelativistic particles. Fix the number of the nonrelativistic particles N . For the mathematical generality, suppose that the dimension of the space in which the nonrelativistic particles move is $d \geq 1$. (This assumption remains throughout this paper.) We use the following identification

$$\mathcal{F}_N \equiv L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_{L^2(\mathbb{R}^d)} \cong L^2(\mathbb{R}^{dN}; \mathcal{F}_{L^2(\mathbb{R}^d)}).$$

For notational simplicity, we write the annihilation or creation operators by $a^\sharp(f)$ instead of $a_{L^2(\mathbb{R}^d)}^\sharp(f)$ in sections 3 and 5. We define a time-zero scalar field $\phi(\hat{f})$ by

$$\phi(\hat{f}) = \frac{1}{\sqrt{2}} \left\{ a^\dagger \left(\frac{\hat{f}}{\sqrt{\omega}} \right) + a \left(\frac{\bar{\hat{f}}}{\sqrt{\omega}} \right) \right\}.$$

Here $\omega = \omega(k) = \sqrt{k^2 + \mu^2}$, $\mu \geq 0$. In this section we require that ϱ is a real valued even function, $\varrho(k) = \varrho(-k)$, with

$$\frac{\hat{\varrho}}{\omega\sqrt{\omega}}, \frac{\hat{\varrho}}{\omega}, \frac{\hat{\varrho}}{\sqrt{\omega}} \in L^2(\mathbb{R}^d). \quad (3.1)$$

For each $x = (x^1, \dots, x^N) \in \mathbb{R}^{dN}$, $x^j \in \mathbb{R}^d$, $j = 1, \dots, N$, we set

$$\tilde{\varrho}(x) = \frac{1}{\sqrt{(2\pi)^d}} \sum_{j=1}^N \hat{\varrho}(k) e^{-ikx^j}.$$

We define

$$H_I(\hat{\varrho}) \equiv \phi(\tilde{\varrho}(\cdot)).$$

For the multiplication operator ω in $L^2(\mathbb{R}^d)$ with the maximal domain, we set $d\Gamma_{L^2(\mathbb{R}^d)}(\omega) \equiv H_b$. Define an operator in \mathcal{F}_N by

$$H_N^\beta(\hat{\varrho}, \Lambda) = -\frac{1}{2m} \Delta_N \otimes I - \Lambda g H_I(\hat{\varrho}) + \Lambda^\beta I \otimes H_b, \quad 1 \leq \beta < \infty,$$

where $g \in \mathbb{R}$ is a coupling constant, $m > 0$ a mass of the nonrelativistic particles, Δ_N the Laplacian in $L^2(\mathbb{R}^{dN})$ and $\Lambda > 0$ a scaling parameter. Moreover we put a decoupled Hamiltonian $H_{\beta,N}(\Lambda)$ by

$$H_N^\beta(\Lambda) = -\frac{1}{2m} \Delta_N \otimes I + \Lambda^\beta I \otimes H_b.$$

We define a class of the set of multiplication operators in $L^2(\mathbb{R}^{dN})$. A multiplication operator V is in a class, $\mathcal{M}_\pm(N)$, if and only if V is infinitesimally small with respect to $-\Delta_N$.

Proposition 3.1 ([2]) *For $\Lambda > 0$ and $V \in \mathcal{M}_\pm(N)$, $H_N^\beta(\hat{\varrho}, \Lambda) + V \otimes I$ is self-adjoint on $D(H_N^\beta(\Lambda))$ and bounded from below. Moreover it is essentially self-adjoint on any core for $H_N^\beta(\Lambda)$.*

In the case of $\beta = 2$, following proposition is well known.

Proposition 3.2 ($[2,3], \beta = 2$) *Let $V \in \mathcal{M}_\pm(N)$. Then*

$$s - \lim_{\Lambda \rightarrow \infty} e^{-it(H_N^\beta(\hat{\rho}, \Lambda) + V \otimes I)} = e^{-it(-\frac{1}{2m}\Delta_N + V + g^2 V(\hat{\rho}))} \otimes P_N.$$

Here P_N is the projection operator onto the subspace in \mathcal{F} , spanned by the vector $\Omega_{L^2(\mathbb{R}^d)}$.

3.2 The case of $\beta = 1$

Put $C_0^\infty(\mathbb{R}^{dN}) \hat{\otimes} \mathcal{F}_{L^2(\mathbb{R}^d)}^\infty \equiv \mathcal{F}_N^\infty$, where $\hat{\otimes}$ denotes the algebraic tensor product. We perform a unitary transformation

$$\mathcal{U}(g) = \exp \left(\frac{g}{\sqrt{2}} \left\{ a^\dagger \left(\frac{\tilde{\rho}(\cdot)}{\omega \sqrt{\omega}} \right) - a \left(\frac{\overline{\tilde{\rho}(\cdot)}}{\omega \sqrt{\omega}} \right) \right\} \right)$$

with the following result:

Proposition 3.3 *The unitary operator $\mathcal{U}(\Lambda^{1-\beta} g)$ maps \mathcal{F}_N^∞ into $D(H_N^\beta(\hat{\rho}, \Lambda))$ with*

$$\begin{aligned} & \mathcal{U}(\Lambda^{1-\beta} g)^{-1} (H_N^\beta(\hat{\rho}, \Lambda) + V \otimes I) \mathcal{U}(\Lambda^{1-\beta} g) \\ &= \frac{1}{2m} \sum_{j=1}^N (\mathbf{p}^j \otimes I - g \Lambda^{1-\beta} \phi_j)^2 + g^2 \Lambda^{2-\beta} V(\hat{\rho}) \otimes I + \Lambda^\beta I \otimes H_b + V \otimes I, \end{aligned} \quad (3.2)$$

on \mathcal{F}_N^∞ , where $\mathbf{p}^j = (-i \frac{\partial}{\partial x_1^j}, \dots, -i \frac{\partial}{\partial x_d^j})$, $\phi_j = (\phi(\hat{\rho}_1^j(\cdot)), \dots, \phi(\hat{\rho}_d^j(\cdot)))$, $j = 1, \dots, N$, and

$$\begin{aligned} \hat{\rho}_\mu^j(x) &= \hat{\rho}_\mu^j(x, k) = \frac{1}{\sqrt{(2\pi)^d}} \frac{\hat{\rho}(k) e^{-ikx^j} k_\mu}{\omega(k)}, \mu = 1, \dots, d, \\ V(\hat{\rho}) &= V(\hat{\rho}, x) = -\frac{1}{2(2\pi)^d} \left\| \sum_{j=1}^N \frac{\hat{\rho} e^{-ikx^j}}{\omega} \right\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Moreover, for sufficiently large $\Lambda > 0$, the right hand side (R.H.S.) of (3.2) is self-adjoint on $D(H_N^\beta(\Lambda))$ and the equation (3.2) can be extended to the equation on $D(H_N^\beta(\Lambda))$.

Proposition 3.3 implies that the following equation holds, for $V \in \mathcal{M}_\pm(N)$ and sufficiently large $\Lambda > 0$;

$$\begin{aligned} & \mathcal{U}(\Lambda^{1-\beta} g)^{-1} (H_N^\beta(\hat{\rho}, \Lambda) - g^2 \Lambda^{2-\beta} V(\hat{\rho}) \otimes I + V \otimes I) \mathcal{U}(\Lambda^{1-\beta} g) \\ &= \frac{1}{2m} \sum_{j=1}^N (\mathbf{p}^j \otimes I - g \Lambda^{1-\beta} \phi_j)^2 + V \otimes I + \Lambda^\beta I \otimes H_b. \end{aligned} \quad (3.3)$$

In this subsection we set $\beta = 1$. Then we define a symmetric operator $Q(\hat{\rho})$, which is independent of Λ , by

$$R.H.S. \text{ of (3.3) } = H_N^\beta(\Lambda) + Q(\hat{\rho}).$$

Lemma 3.4 *Let $V \in \mathcal{M}_\pm(N)$. Then, for any $\epsilon > 0$, there exists Λ_0 and $b(\epsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(Q(\hat{\rho})) \supset D(H_N^\beta(\Lambda))$ with*

$$\|Q(\hat{\rho})\Phi\|_{\mathcal{F}_N} \leq \epsilon \|H_N^\beta(\Lambda)\Phi\|_{\mathcal{F}_N} + b(\epsilon) \|\Phi\|_{\mathcal{F}_N}, \Phi \in D(H_F). \quad (3.4)$$

Moreover $D(Q(\hat{\rho})) \supset D(-\Delta_N) \hat{\otimes} \text{Ker} H_b$ with, for $z \in \mathbb{C} \setminus [0, \infty)$,

$$s - \lim_{\Lambda \rightarrow \infty} Q(\hat{\rho}) (H_N^\beta(\Lambda) - z)^{-1} = Q(\hat{\rho}) \left[\left(-\frac{1}{2m} \Delta_N - z \right)^{-1} \otimes P_N \right]. \quad (3.5)$$

Proof: The proof of (3.4) follows from fundamental estimates with respect to a^\sharp and H_b .

By (3.4), for any $\epsilon > 0$, taking sufficiently large $\Lambda > 0$, we see that

$$\begin{aligned} & \left\| Q(\hat{\rho}) \left\{ (H_N^\beta(\Lambda) - z)^{-1} - \left(-\frac{1}{2m} \Delta_N - z \right)^{-1} \otimes P_N \right\} \Phi \right\|_{\mathcal{F}_N} \\ & \leq \epsilon \left\| H_N^\beta(\Lambda) \left\{ (H_N^\beta(\Lambda) - z)^{-1} - \left(-\frac{1}{2m} \Delta_N - z \right)^{-1} \otimes P_N \right\} \Phi \right\|_{\mathcal{F}_N} \\ & \quad + b(\epsilon) \left\| \left\{ (H_N^\beta(\Lambda) - z)^{-1} - \left(-\frac{1}{2m} \Delta_N - z \right)^{-1} \otimes P_N \right\} \Phi \right\|_{\mathcal{F}_N}. \end{aligned}$$

Taking $\Lambda \rightarrow \infty$ on the both sides above, we have

$$\begin{aligned} & \lim_{\Lambda \rightarrow \infty} \left\| Q(\hat{\rho}) \left\{ (H_N^\beta(\Lambda) - z)^{-1} - \left(-\frac{1}{2m} \Delta_N - z \right)^{-1} \otimes P_N \right\} \Phi \right\|_{\mathcal{F}_N} \\ & \leq \epsilon \|(I \otimes I - I \otimes P_N) \Phi\|_{\mathcal{F}_N}. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, (3.5) follows. \square

Theorem 3.5 ($\beta = 1$) *Let $V \in \mathcal{M}_\pm(N)$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Put $\omega_0 = \omega_0(k) = |k|$ and*

$$\delta(\hat{\rho}) = \frac{1}{2(2\pi)^d} \left\| \frac{\hat{\rho}\omega_0}{\omega\sqrt{\omega}} \right\|_{L^2(\mathbb{R}^d)}^2.$$

Then

$$\begin{aligned} & s - \lim_{\Lambda \rightarrow \infty} (H_N^\beta(\hat{\rho}, \Lambda) - g^2 \Lambda^{2-\beta} V(\hat{\rho}) \otimes I + V \otimes I - z)^{-1} \\ & = \mathcal{U}(g) \left\{ \left(-\frac{1}{2m} \Delta_N + g^2 N \delta(\hat{\rho}) + V - z \right)^{-1} \otimes P_N \right\} \mathcal{U}(g)^{-1}. \end{aligned} \quad (3.6)$$

Proof: From (3.3) it follows that

$$\begin{aligned} & \left(H_N^\beta(\hat{\rho}, \Lambda) - g^2 \Lambda^{2-\beta} V(\hat{\rho}) \otimes I + V \otimes I - z \right)^{-1} \\ &= \mathcal{U}(\Lambda^{1-\beta} g) \left(H_N^\beta(\Lambda) + Q(\hat{\rho}) - z \right)^{-1} \mathcal{U}(\Lambda^{1-\beta} g)^{-1}. \end{aligned}$$

By the fact that $\mathcal{U}(\Lambda^{1-\beta} g)$ is independent of Λ , it is enough to show that

$$s - \lim_{\Lambda \rightarrow \infty} \left(H_N^\beta(\Lambda) + Q(\hat{\rho}) - z \right)^{-1} = \left(-\frac{1}{2m} \Delta_N + g^2 N \delta(\hat{\rho}) + V - z \right)^{-1} \otimes P_N.$$

Since the partial expectation of $Q(\hat{\rho})$ with respect to $\Omega_{L^2(\mathbb{R}^d)}$ is

$$\begin{aligned} E_{\Omega_{L^2(\mathbb{R}^d)}}(Q(\hat{\rho})) &= g^2 \sum_{j=1}^N \sum_{\mu=1}^3 \frac{1}{2} \left\| \frac{\tilde{\rho}_\mu^j(\cdot)}{\sqrt{\omega}} \right\|_{L^2(\mathbb{R}^d)}^2 + V \\ &= g^2 N \delta(\hat{\rho}) + V, \end{aligned}$$

it follows (3.6) from Lemma 3.4 and Proposition 2.1 with the following correspondence :

$$A = -\frac{1}{2m} \Delta_N, \quad B = H_b, \quad C_\Lambda = C = Q(\hat{\rho}), \quad G = \Omega_{L^2(\mathbb{R}^d)}.$$

□

3.3 The case of $1 < \beta < 2$, $2 < \beta$

First we study the case of $1 < \beta < 2$. We put the R.H.S. of (3.3) by

$$R.H.S. of (3.3) = H_N^\beta(\Lambda) + Q^1(\hat{\rho}, \Lambda). \quad (3.7)$$

Similar to (3.4) and (3.5), one can see that, for $V \in \mathcal{M}_\pm(N)$ and any $\epsilon > 0$, there exists Λ_0 and $b(\epsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(Q^1(\hat{\rho}, \Lambda)) \supset D(H_N^\beta(\Lambda))$ with

$$\left\| Q^1(\hat{\rho}, \Lambda) \Phi \right\|_{\mathcal{F}_N} \leq \epsilon \left\| H_N^\beta(\Lambda) \Phi \right\|_{\mathcal{F}_N} + b(\epsilon) \left\| \Phi \right\|_{\mathcal{F}_N}, \quad \Phi \in D(H_N^\beta(\Lambda)). \quad (3.8)$$

Moreover $D(Q^1(\hat{\rho}, \Lambda)) \supset D(-\Delta_N) \hat{\otimes} \text{Ker} H_b$ and, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s - \lim_{\Lambda \rightarrow \infty} Q^1(\hat{\rho}, \Lambda) \left(H_N^\beta(\Lambda) - z \right)^{-1} = \left[V \left(-\frac{1}{2m} \Delta_N - z \right)^{-1} \right] \otimes P_N. \quad (3.9)$$

Note that, for $\beta > 1$,

$$s - \lim_{\Lambda \rightarrow \infty} \mathcal{U} \left(\frac{g}{\Lambda^{\beta-1}} \right) = I. \quad (3.10)$$

Hence we prove the following theorem

Theorem 3.6 ($1 < \beta < 2$) Let $V \in \mathcal{M}_\pm(N)$, $z \in \mathbb{C} \setminus \mathbb{R}$. Then

$$s - \lim_{\Lambda \rightarrow \infty} \left(H_N^\beta(\hat{\rho}, \Lambda) - g^2 \Lambda^{2-\beta} V(\hat{\rho}) \otimes I + V \otimes I - z \right)^{-1} = \left(-\frac{1}{2m} \Delta_N + V - z \right)^{-1} \otimes P_N.$$

Proof: Since the partial expectation of $V \otimes I$ with respect to $\Omega_{L^2(\mathbb{R}^d)}$ is $E_{\Omega_{L^2(\mathbb{R}^d)}}(V \otimes I) = V$, from (3.8), (3.9), (3.10) and Proposition 2.1, theorem follows with the following correspondence:

$$A = -\frac{1}{2m} \Delta_N, \quad B = H_b, \quad C_\Lambda = Q^1(\hat{\rho}, \Lambda), \quad C = V \otimes I \quad G = \Omega_{L^2(\mathbb{R}^d)}.$$

□

Secondly we study the case of $2 < \beta$. In this case, note that we do not need to subtract the renormalization $\Lambda^{2-\beta} V(\hat{\rho}) \otimes I$. Put the R.H.S. of (3.2) by

$$R.H.S. \text{ of (3.2) } = H_N^\beta(\Lambda) + Q^2(\hat{\rho}, \Lambda).$$

By the same argument as that of the case of $1 < \beta < 2$ with $Q^1(\Lambda, \hat{\rho})$ replaced by $Q^2(\Lambda, \hat{\rho})$, one can easily prove the following theorem.

Theorem 3.7 ($2 < \beta$) Let $V \in \mathcal{M}_\pm(N)$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then

$$s - \lim_{\Lambda \rightarrow \infty} \left(H_N^\beta(\hat{\rho}, \Lambda) + V \otimes I - z \right)^{-1} = \left(-\frac{1}{2m} \Delta_N + V - z \right)^{-1} \otimes P_N.$$

4 THE PAULI-FIERZ MODEL

4.1 The Pauli-Fierz model

In this section, we study the Pauli-Fierz model in quantum electrodynamics with the dipole approximation. Let

$$\mathcal{W} = \underbrace{L^2(\mathbb{R}^d) \oplus \dots \oplus L^2(\mathbb{R}^d)}_{d-1}.$$

For $(0 \oplus \dots \underbrace{f}_{\text{the } r\text{-th}} \dots \oplus 0) \in \mathcal{W}$, we set $a_{\mathcal{W}}^\sharp(0 \oplus \dots \underbrace{f}_{\text{the } r\text{-th}} \dots \oplus 0) = a^{\sharp(r)}(f)$. We write

$$a_{\mathcal{W}}^{\sharp(r)}(g) = \int a_{\mathcal{W}}^{\sharp(r)}(k) g(k) dk, \quad r = 1, \dots, d-1.$$

Let $e^r : \mathbb{R}^d \rightarrow \mathbb{R}^d, r = 1, \dots, d-1$, be measurable functions so that

$$(1) e^r(k) \cdot k = 0, \quad r = 1, \dots, d-1, \quad (2) e^r(k) \cdot e^s(k) = \delta_{rs}.$$

We denote the μ -th component of e^r by $e_\mu^r, \mu = 1, \dots, d$. The quantized smeared radiation field $A_\mu(\hat{f}, x)$ with f in the Coulomb gauge, and the conjugate momentum $\Pi_\mu(\hat{f}, x), \mu = 1, \dots, d, x \in \mathbb{R}^d$, are defined by

$$A_\mu(\hat{f}, x) = \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \int \left\{ a_{\mathcal{W}}^{(r)\dagger}(k) \frac{e_\mu^r(k) \tilde{f}(k) e^{-ikx}}{\sqrt{\omega(k)}} + a_{\mathcal{W}}^{(r)}(k) \frac{e_\mu^r(k) \hat{f}(k) e^{ikx}}{\sqrt{\omega(k)}} \right\} dk,$$

$$\Pi_\mu(\hat{f}, x) = \frac{i}{\sqrt{2}} \sum_{r=1}^{d-1} \int \left\{ a_{\mathcal{W}}^{(r)\dagger}(k) \sqrt{\omega(k)} e_\mu^r(k) \tilde{f}(k) e^{-ikx} - a_{\mathcal{W}}^{(r)}(k) \sqrt{\omega(k)} e_\mu^r(k) \hat{f}(k) e^{ikx} \right\} dk.$$

Here $\tilde{g}(k) = g(-k)$. We define the free Hamiltonian in $\mathcal{F}_{\mathcal{W}}$ by

$$d\Gamma_{\mathcal{W}}(\underbrace{\omega \oplus \dots \oplus \omega}_{d-1}) = H_{EM}.$$

We require that $\hat{\varrho}$ satisfies (3.1) and ϱ is real-valued rotation invariant function throughout this section. Then the Pauli-Fierz Hamiltonian with the ultraviolet cut-off function $\hat{\varrho}$ and with N -nonrelativistic particles is defined as an operator acting in

$$\mathcal{L}_N = L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_{\mathcal{W}} \cong L^2(\mathbb{R}^{dN}; \mathcal{F}_{\mathcal{W}}),$$

by

$$\frac{1}{2m} \sum_{j=1}^N (\mathbf{p}^j \otimes I - eA(\hat{\varrho}, \cdot))^2 + I \otimes H_{EM}, \quad (4.1)$$

where $e \in \mathbb{R}$ is a coupling constant and $A(\hat{\varrho}, \cdot) = (A_1(\hat{\varrho}, \cdot), \dots, A_d(\hat{\varrho}, \cdot))$. Introducing the polarization vector e^r , which corresponds to taking the Coulomb gauge, we see that, on a suitable dense domain,

$$[\mathbf{p}^j \otimes I, A(\hat{\varrho}, \cdot)] = 0.$$

Then formally we may rewrite (4.1) by

$$-\frac{1}{2m} \Delta_N \otimes I - \frac{e}{m} \sum_{j=1}^N \sum_{\mu=1}^d (\mathbf{p}_\mu^j \otimes I) A_\mu(\hat{\varrho}, \cdot) + \frac{e^2 N}{2m} A^2(\hat{\varrho}, \cdot) + I \otimes H_{EM}.$$

Here, for simplicity, we introduce the following assumptions to the Pauli-Fierz Hamiltonian:

(1) The self-interaction term $A^2(\hat{\rho}, \cdot)$ is neglected.

(2) We introduce the dipole approximation, i.e., $A(\hat{\rho}, x)$ is replaced by $A(\hat{\rho}, 0)$.

Then, putting $A(\hat{\rho}, 0) = I \otimes A(\hat{\rho})$, our Hamiltonian is as follows:

$$H_{EM}^\beta(\hat{\rho}, \Lambda) = -\frac{1}{2m}\Delta_N \otimes I - \Lambda e H_I^{EM}(\hat{\rho}) + \Lambda^\beta I \otimes H_{EM},$$

where

$$H_I^{EM}(\hat{\rho}) = \frac{1}{m} \sum_{j=1}^N \sum_{\mu=1}^d \mathbf{p}_\mu^j \otimes A_\mu(\hat{\rho}).$$

Put

$$H_{EM}^\beta(\Lambda) = -\frac{1}{2m}\Delta_N \otimes I + \Lambda^\beta I \otimes H_{EM}.$$

Theorem 4.1 ([1]) *Let $V \in \mathcal{M}_\pm(N)$. Then the operator $H_{EM}^\beta(\hat{\rho}, \Lambda) + V \otimes I$ is self-adjoint on $D(H_{EM}^\beta(\Lambda))$ and bounded from below. Moreover it is essentially self-adjoint on any core for $H_{EM}^\beta(\Lambda)$.*

We define a unitary operator by

$$S(e) = \exp \left(-ie \left(\sum_{j=1}^N \sum_{\mu=1}^d \frac{1}{m} \mathbf{p}_\mu^j \otimes \Pi_\mu \left(\frac{\hat{\rho}}{\omega^2} \right) \right) \right),$$

where we put $\Pi_\mu(0, f) = I \otimes \Pi_\mu(f)$.

Lemma 4.2 ([1]) *Let $V \in \mathcal{M}_\pm(N)$. Then the unitary operator $S(e)$ maps $D(H_{EM}^\beta(\hat{\rho}, \Lambda))$ onto itself with*

$$\begin{aligned} & S(\Lambda^{1-\beta} e)^{-1} (H_{EM}^\beta(\hat{\rho}, \Lambda) + V \otimes I) S(\Lambda^{1-\beta} e) \\ &= - \left(\frac{1}{2m} + \Lambda^{2-\beta} \frac{e^2}{2M} \right) \Delta_N \otimes I + \Lambda^\beta I \otimes H_{EM} + V_\beta(\hat{\rho}, \Lambda), \end{aligned} \tag{4.2}$$

where

$$\frac{1}{2M} = \frac{d-1}{d} \left(\frac{1}{m} \right)^2 \left\| \frac{\hat{\rho}}{\omega} \right\|_{L^2(\mathbb{R}^d)}^2, \quad V_\beta(\hat{\rho}, \Lambda) = S(\Lambda^{1-\beta} e)^{-1} (V \otimes I) S(\Lambda^{1-\beta} e).$$

To obtain the scaling limit of the case of $\beta = 1$, we need to fix a $dN \times dN$ -matrix \mathbf{T} so that

$$\mathbf{T} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & 1 & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \mathbf{T}^{-1} = \begin{pmatrix} N & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then, for a multiplication operator V in $L^2(\mathbb{R}^{dN})$, we put

$$V_{eff}^{\hat{\rho}}(x) = (2\pi C_N(\hat{\rho}))^{-\frac{d}{2}} \int_{\mathbb{R}^d} dy V \left(\mathbf{T}^{-1}(y, (\mathbf{T}x)_2, \dots, (\mathbf{T}x)_N) \right) e^{-\frac{|(\mathbf{T}x)_1 - y|^2}{2C_N(\hat{\rho})}},$$

where

$$C_N(\hat{\rho}) = \frac{d-1}{2d} \left(\frac{e}{m} \right)^2 \int_{\mathbb{R}^{dN}} dk \frac{|\hat{\rho}(k)|^2}{\omega(k)^3},$$

and $(\mathbf{T}x)_j \in \mathbb{R}^d, j = 1, \dots, N$, denotes the j -th element of $\mathbf{T}x \in \mathbb{R}^{dN}$. In the case of $\beta = 1$, the following proposition is well known.

Proposition 4.3 ([1,6,7], $\beta = 1$) *Let $V \in \mathcal{M}_{\pm}(N)$ with*

$$|V|_{eff}^{\hat{\rho}}(x) < \infty, a.e. x \in \mathbb{R}^{dN}, \quad |V|_{eff}^{\hat{\rho}} \in L_{loc}^1(\mathbb{R}^{dN}).$$

Then $-\frac{1}{2m}\Delta_N + V_{eff}^{\hat{\rho}}$ is self-adjoint on $D(-\Delta_N)$ with, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned} & s - \lim_{\Lambda \rightarrow \infty} \left(H_{EM}^{\beta}(\hat{\rho}, \Lambda) + V \otimes I + \Lambda^{2-\beta} \frac{e^2}{2M} \Delta_N \otimes I - z \right)^{-1} \\ &= S(e) \left\{ \left(-\frac{1}{2m} \Delta_N + V_{eff}^{\hat{\rho}} - z \right)^{-1} \otimes P_{EM} \right\} S(e)^{-1}, \end{aligned}$$

where P_{EM} is the projection operator onto the subspace $\{k\Omega_{\mathcal{W}} | k \in \mathbb{C}\} \subset \mathcal{F}_{\mathcal{W}}$.

4.2 The case of $\beta = 2$

Put

$$\widetilde{H_{EM}^{\beta}}(\Lambda) = - \left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N \otimes I + \Lambda^{\beta} I \otimes H_{EM}.$$

Lemma 4.4 Let $V \in \mathcal{M}_\pm(N)$. Then, for any $\epsilon > 0$, there exists Λ_0 and $b(\epsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(V_\beta(\hat{\rho}, \Lambda)) \supset D(\widetilde{H_{EM}^\beta}(\Lambda))$ with

$$\|V_\beta(\hat{\rho})\Phi\|_{\mathcal{L}_N} \leq \epsilon \left\| \widetilde{H_{EM}^\beta}(\Lambda)\Phi \right\|_{\mathcal{L}_N} + b(\epsilon) \|\Phi\|_{\mathcal{L}_N}, \Phi \in D(\widetilde{H_{EM}^\beta}(\Lambda)). \quad (4.3)$$

Moreover, for $z \in \mathbb{C} \setminus [0, \infty)$,

$$s - \lim_{\Lambda \rightarrow \infty} V_\beta(\hat{\rho}, \Lambda) \left(\widetilde{H_{EM}^\beta}(\Lambda) - z \right)^{-1} = \left[V \left(- \left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N - z \right)^{-1} \right] \otimes P_{EM}. \quad (4.4)$$

Proof: Since V is infinitesimally small with respect to $-\Delta_N$ and $-\Delta_N$ commutes $S(e)$, one can derive (4.3). Put $\left(\widetilde{H_{EM}^\beta}(\Lambda) - z \right)^{-1} = K_\Lambda$, $\left(- \left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N - z \right)^{-1} \otimes P_{EM} = K_\infty$ and $S(\Lambda^{1-\beta}e) = S_\Lambda$. Note that, for $\beta > 1$,

$$s - \lim_{\Lambda \rightarrow \infty} S(\Lambda^{1-\beta}e) = I,$$

By (4.3), for any $\epsilon > 0$, taking sufficiently large $\Lambda > 0$, we have

$$\begin{aligned} & \|V_\beta(\hat{\rho}, \Lambda)K_\Lambda\Phi - (V \otimes I)K_\infty\Phi\| \\ & \leq \epsilon \|-\Delta_N(K_\Lambda - K_\infty)\Phi\|_{\mathcal{L}_N} + \epsilon \|-\Delta_N(S_\Lambda K_\infty - K_\infty)\Phi\|_{\mathcal{L}_N} \\ & + b(\epsilon) \|(K_\Lambda - K_\infty)\Phi\|_{\mathcal{L}_N} + b(\epsilon) \|(S_\Lambda K_\infty - K_\infty)\Phi\|_{\mathcal{L}_N} + \|(S_\Lambda^{-1} - I)(V \otimes I)K_\infty\Phi\|_{\mathcal{L}_N}. \end{aligned}$$

Taking $\Lambda \rightarrow \infty$ on the both sides above, we have

$$\lim_{\Lambda \rightarrow \infty} \|V_\beta(\hat{\rho}, \Lambda)K_\Lambda\Phi - (V \otimes I)K_\infty\Phi\|_{\mathcal{L}_N} \leq \epsilon \|(I \otimes I - I \otimes P_{EM})\Phi\|_{\mathcal{L}_N}.$$

Since $\epsilon > 0$ is arbitrary, (4.4) follows. \square

Theorem 4.5 ($\beta = 2$) Let $V \in \mathcal{M}_\pm(N)$. Then, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s - \lim_{\Lambda \rightarrow \infty} \left(\widetilde{H_{EM}^\beta}(\hat{\rho}, \Lambda) + V \otimes I - z \right)^{-1} = \left\{ - \left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N + V - z \right\}^{-1} \otimes P_{EM}. \quad (4.5)$$

Proof: By virtue of (4.2), we see that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\left(\widetilde{H_{EM}^\beta}(\hat{\rho}, \Lambda) + V \otimes I - z \right)^{-1} = S(\Lambda^{1-\beta}e) \left(\widetilde{H_{EM}^\beta}(\Lambda) + V_\beta(\hat{\rho}, \Lambda) - z \right)^{-1} S(\Lambda^{1-\beta}e)^{-1},$$

and the partial expectation of $V \otimes I$ with respect to $\Omega_{\mathcal{W}}$ is $E_{\Omega_{\mathcal{W}}}(V \otimes I) = V$. Hence, it follows (4.5) from Lemma 4.4 and Proposition 2.1 with the following correspondence:

$$A = - \left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N, B = H_{EM}, C(\Lambda) = V_\beta(\hat{\rho}, \Lambda), C = V \otimes I, G = \Omega_{\mathcal{W}}.$$

\square

4.3 The case of $1 < \beta < 2$, $2 < \beta$

For the case of $1 < \beta < 2$, by (4.2), we should subtract the term $-\Lambda^{2-\beta} \frac{e^2}{2M} \Delta_N \otimes I$ from the original Hamiltonian $H_{EM}^\beta(\hat{\rho}, \Lambda)$, and for the case of $\beta > 2$, we do not need any renormalization. Hence, the similar argument of the cases of $\beta = 2$ and $\beta = 1$ gives an asymptotic behaviors of $H_{EM}^\beta(\hat{\rho}, \Lambda)$. See Fig. 6.2.

5 THE SPIN-BOSON MODEL

5.1 The spin-boson model

In this section we study the spin-boson model. The total Hamiltonian of the spin-boson model is defined as an operator acting in the Hilbert space

$$\mathcal{L}_{SB} = \mathbb{C}^2 \otimes \mathcal{F}_{L^2(\mathbb{R}^d)} \cong \mathcal{F}_{L^2(\mathbb{R}^d)} \oplus \mathcal{F}_{L^2(\mathbb{R}^d)},$$

by

$$H_{SB}^\beta(\lambda, \Lambda) = \nu \sigma_1 + \Lambda \sigma_3 \otimes (a^\dagger(\bar{\lambda}) + a(\lambda)) + \Lambda^\beta I \otimes H_{SB}.$$

Here $H_{SB} = d\Gamma_{L^2(\mathbb{R}^d)}(\omega)$, $\nu > 0$, $\lambda \in L^2(\mathbb{R}^d)$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In what follows, we assume that

$$\lambda, \frac{\lambda}{\sqrt{\omega}}, \frac{\lambda}{\omega} \in L^2(\mathbb{R}^d).$$

Theorem 5.1 ([1]) *The operator $H_{SB}(\lambda, \Lambda)$ is self-adjoint on $D(I \otimes H_{SB})$ and bounded from below. Moreover essentially self-adjoint on any core for $I \otimes H_{SB}$.*

We define a unitary operator by

$$\mathbf{T}(\lambda) = \begin{pmatrix} e^{+\{a^\dagger(\frac{\lambda}{\omega}) - a(\frac{\lambda}{\omega})\}} & 0 \\ 0 & e^{-\{a^\dagger(\frac{\lambda}{\omega}) - a(\frac{\lambda}{\omega})\}} \end{pmatrix} \equiv \begin{pmatrix} T_+(\lambda) & 0 \\ 0 & T_-(\lambda) \end{pmatrix}.$$

In the case of $\beta = 1$, following proposition is well known.

Proposition 5.2 ([1], $\beta = 1$) Let $F(\lambda) = \langle \Omega_{L^2(\mathbb{R}^d)}, T_+(\lambda) \Omega_{L^2(\mathbb{R}^d)} \rangle_{\mathcal{F}_{L^2(\mathbb{R}^d)}}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then

$$s - \lim_{\Lambda \rightarrow \infty} (H_{SB}^\beta(\lambda, \Lambda) - \Lambda^{2-\beta} E_{SB} - z)^{-1} = \mathbf{T}(\lambda) \{ (\nu F(\lambda) \sigma_1 - z)^{-1} \otimes P_{SB} \} \mathbf{T}(\lambda)^{-1}.$$

Here P_{SB} is the projection operator onto $\{k\Omega | k \in \mathbb{C}\} \subset \mathcal{F}_{L^2(\mathbb{R}^d)}$ and $E_{SB} = -\left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2(\mathbb{R}^d)}^2$.

5.2 the case of $\beta = 2$

It is well known and easily checked that $\mathbf{T}(\Lambda^{1-\beta}\lambda)$ maps $D(I \otimes H_{SB})$ onto itself with

$$\begin{aligned} & \mathbf{T}^{-1}(\Lambda^{1-\beta}\lambda) H_{SB}^\beta(\lambda) \mathbf{T}(\Lambda^{1-\beta}\lambda) \\ &= \nu \begin{pmatrix} 0 & T_-^2(\Lambda^{1-\beta}\lambda) \\ T_+^2(\Lambda^{1-\beta}\lambda) & 0 \end{pmatrix} + \Lambda^\beta I \otimes H_{SB} + \Lambda^{2-\beta} E_{SB}. \end{aligned} \quad (5.1)$$

Theorem 5.3 ($\beta = 2$) Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then

$$s - \lim_{\Lambda \rightarrow \infty} (H_{SB}^\beta(\lambda, \Lambda) - z)^{-1} = (\nu \sigma_1 + E_{SB} - z)^{-1} \otimes P_{SB}. \quad (5.2)$$

Proof: We see that, by (5.1),

$$\begin{aligned} & \mathbf{T}^{-1}(\Lambda^{1-\beta}\lambda) (H_{SB}^\beta(\lambda, \Lambda) - z)^{-1} \mathbf{T}(\Lambda^{1-\beta}\lambda) \\ &= \left\{ \nu \begin{pmatrix} 0 & T_-^2(\Lambda^{1-\beta}\lambda) \\ T_+^2(\Lambda^{1-\beta}\lambda) & 0 \end{pmatrix} + \Lambda^\beta I \otimes H_{SB} + \Lambda^{2-\beta} E_{SB} - z \right\}^{-1}. \end{aligned}$$

It is easily seen that $s - \lim_{\Lambda \rightarrow \infty} \mathbf{T}(\Lambda^{1-\beta}\lambda) = I$ and

$$\begin{aligned} & s - \lim_{\Lambda \rightarrow \infty} \nu \begin{pmatrix} 0 & T_-^2(\Lambda^{1-\beta}\lambda) \\ T_+^2(\Lambda^{1-\beta}\lambda) & 0 \end{pmatrix} (\Lambda^{2-\beta} E_{SB} + \Lambda^\beta I \otimes H_{SB} - z)^{-1} \\ &= [\nu \sigma_1 (E_{SB} - z)^{-1}] \otimes P_{SB}. \end{aligned}$$

Hence, with the following correspondence:

$$A = E_{SB}, B = H_{SB}, C(\Lambda) = \nu \begin{pmatrix} 0 & T_-^2(\Lambda^{1-\beta}\lambda) \\ T_+^2(\Lambda^{1-\beta}\lambda) & 0 \end{pmatrix}, C = \nu \sigma_1, G = \Omega_{L^2(\mathbb{R}^d)},$$

one can easily check the conditions with respect to $C(\Lambda)$ and C in section 2. Since the partial expectation of $\nu \sigma_1 \otimes I$ with respect to $\Omega_{L^2(\mathbb{R}^d)}$ is $E_{\Omega_{L^2(\mathbb{R}^d)}}(\nu \sigma_1 \otimes I) = \nu \sigma_1$, we get (5.2) by Proposition 2.1. \square

5.3 The case of $1 < \beta < 2, 2 < \beta$

For the case of $1 < \beta < 2$, by (5.1), we should subtract the term $\Lambda^{2-\beta} E_{SB}$ from the original Hamiltonian $H_{SB}^\beta(\lambda, \Lambda)$, and for the case of $\beta > 2$, we do not need any renormalization. Hence, the similar argument of the cases of $\beta = 2$ and $\beta = 1$ gives an asymptotic behaviors of $H_{SB}^\beta(\lambda, \Lambda)$. See Fig 6.3.

6 CONCLUDING REMARKS

(1) In section 4, we studied the Pauli-Fierz model neglected the terms $A^2(\hat{p}, \cdot)$. In [6,7], we studied the Pauli-Fierz Hamiltonian with the terms $A^2(\hat{p}, \cdot)$. By the same method developed in [6,7], we can investigate the following scaling Hamiltonians:

$$-\frac{1}{2m}\Delta_N \otimes I - \Lambda e H_I^{EM}(\hat{p}) + \Lambda I \otimes H_{EM} + \frac{e^2 N}{2m} A^2(\hat{p}, \cdot) + V \otimes I, \quad (6.1)$$

$$-\frac{1}{2m}\Delta_N \otimes I - \Lambda e H_I^{EM}(\hat{p}) + \Lambda^2 I \otimes H_{EM} + \Lambda^2 \frac{e^2 N}{2m} A^2(\hat{p}, \cdot) + V \otimes I. \quad (6.2)$$

Introducing different renormalizations from those given in this paper, we can get effective Hamiltonians of (6.1) and (6.2).

(2) In the case of $0 < \beta < 1$, we need delicate discussions of asymptotic behaviors of unitary operators $\mathcal{U}(\Lambda^{1-\beta}g)$, $S(\Lambda^{1-\beta}e)$ and $\mathbf{T}(\Lambda^{1-\beta}\lambda)$ as $\Lambda \rightarrow \infty$. We omit the discussions.

	Effective Hamiltonian	Unitary operator	Renormalization
$\beta > 2$	$-\frac{1}{2m}\Delta_N + V$	I	0
$\beta = 2$	$-\frac{1}{2m}\Delta_N + g^2 V(\hat{p}) + V$	I	0
$1 < \beta < 2$	$-\frac{1}{2m}\Delta_N + V$	I	$g^2 \Lambda^{2-\beta} V(\hat{p})$
$\beta = 1$	$-\frac{1}{2m}\Delta_N + g^2 N \delta(\hat{p}) + V$	$\mathcal{U}(g)$	$g^2 \Lambda V(\hat{p})$

Fig 6.1 β -coupling Nelson model $H_N^\beta(\hat{p}, \Lambda)$

	Effective Hamiltonian	Unitary operator	Renormalization
$\beta > 2$	$-\frac{1}{2m}\Delta_N + V$	I	0
$\beta = 2$	$-\left(\frac{1}{2m} + \frac{e^2}{2M}\right)\Delta_N + V$	I	0
$1 < \beta < 2$	$-\frac{1}{2m}\Delta_N + V$	I	$-\Lambda^{2-\beta}\frac{e^2}{2M}\Delta_N$
$\beta = 1$	$-\frac{1}{2m}\Delta_N + V_{eff}^{\hat{e}}$	$S(e)$	$-\Lambda\frac{e^2}{2M}\Delta_N$

Fig.6.2 β -coupling Pauli-Fierz model $H_{EM}^{\beta}(\hat{e}, \Lambda)$

	Effective Hamiltonian	Unitary operator	Renormalization
$\beta > 2$	$\nu\sigma_1$	I	0
$\beta = 2$	$\nu\sigma_1 + E_{SB}$	I	0
$1 < \beta < 2$	$\nu\sigma_1$	I	$\Lambda^{2-\beta}E_{SB}$
$\beta = 1$	$\nu F(\lambda)\sigma_1$	$\mathbf{T}(\lambda)$	ΛE_{SB}

Fig.6.3 β -coupling spin-boson model $H_{SB}^{\beta}(\lambda, \Lambda)$

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